Complex Analysis: Final Exam

MartiniPlaza, Wednesday 30 January 2019, 14:00–17:00
Exam duration: 3 hours

Instructions — read carefully before starting

- Write very clearly your **full name** and **student number** on the envelope and at the top of each answer sheet.
- Use the ruled paper for writing the answers and use the blank paper as scratch paper. After finishing put your answers in the envelope. **Do NOT seal the envelope!** You must return the scratch paper and the printed exam (separately from the envelope). The exam and its solutions will be uploaded to Nestor in the following days.
- Solutions should be complete and clearly present your reasoning. **When you use known results (lemmas, theorems, formulas, etc.) you must explicitly state and verify the corresponding conditions.**
- 10 points are "free". There are 6 questions and the maximum number of points is 100. The exam grade is the total number of points divided by 10.

Question 1 (10 points)

Show that if $f(z)$ and $\overline{f(z)}$ are both analytic in a domain D then $f(z)$ is constant in D.

Solution

Let

$$f(z) = u(x, y) + \mathrm{i}v(x, y).$$

Then

$$\overline{f(z)} = u(x, y) - \mathrm{i}v(x, y).$$

Since both of these functions are analytic, the Cauchy-Riemann equations hold for every $(x, y) \in D$. Therefore, for $f(z)$ we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

while for $\overline{f(z)}$ we have

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

Combining equations we obtain

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0,$$

and consequently also

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0.$$

Since D is a domain and the partial derivatives of u and v are zero we conclude that these functions are constant within D and thus $f = u + \mathrm{i}v$ is also constant.

Question 2 (20 points)

(a) (8 points) Consider the integral

$$\mathrm{pv}\int_{-\infty}^{\infty} \frac{e^{-2\mathrm{i}x}}{x+1}\,\mathrm{d}x.$$

Specify and draw the (closed) contour that you should use to compute such an integral with the calculus of residues. Give full justification for your choice of contour.

Solution

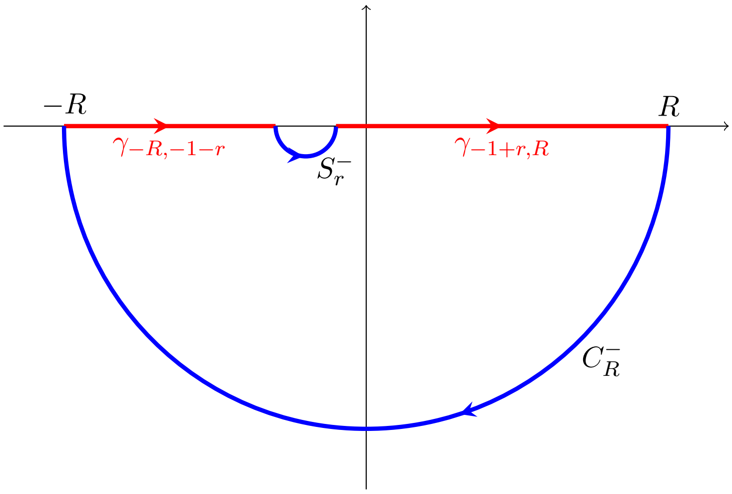
There are two issues we have to consider here. First, the factor $e^{-2\mathrm{i}x}$ in the integral and, second, the fact that the integrand has a pole of order 1 at $x = -1$.

Here we consider the contour

$$\Gamma = \gamma_{-R,-1-r} + S_r^- + \gamma_{-1+r,R} + C_R^-,$$

where $\gamma_{-R,-1-r}$ is the straight line connecting $-R$ to $-1-r$ on the real axis, $\gamma_{-1+r,R}$ is the straight line connecting $-1+r$ to R, S_r^- is the half-circle centered at -1 and connecting $-1-r$ to $-1+r$ in the lower half-plane (although we could have chosen S_r^+ in the upper half-plane) and C_R^- is the half-circle centered at 0 and connecting R to $-R$ in the lower half-plane.

We need S_r^- (or S_r^+) to bypass the singularity of the integrand at -1 and we need to take C_R^- instead of C_R^+ so that we can apply Jordan's lemma for $e^{-2\mathrm{i}x}$ and estimate that the contribution to the integral from C_R^- goes to 0 as $R \to \infty$.



(b) (12 points) Evaluate the integral

$$\mathrm{pv}\int_{-\infty}^{\infty} \frac{x+1}{(x^2+1)^2}\,\mathrm{d}x,$$

using the calculus of residues. Give complete arguments.

Solution

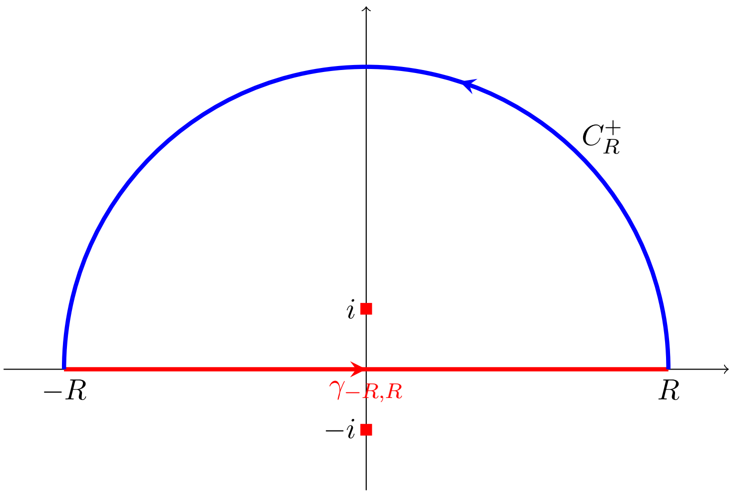
Let

$$I = \mathrm{pv}\int_{-\infty}^{\infty} \frac{x+1}{(x^2+1)^2}\,\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} \frac{x+1}{(x^2+1)^2}\,\mathrm{d}x.$$

The denominator of the integrand, $(x^2+1)^2$, factorizes as $(x-\mathrm{i})^2(x+\mathrm{i})^2$. Therefore, $\pm\mathrm{i}$ are poles of order 2 of the integrand. For the integration we consider the contour

$$\Gamma = \gamma_{-R,R} + C_R^+,$$

shown below.



Then we have

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_{\gamma_{-R,R}} f(z)\,\mathrm{d}z + \int_{C_R^+} f(z)\,\mathrm{d}z,$$

where

$$f(z) = \frac{z+1}{(z^2+1)^2} = \frac{z+1}{(z-\mathrm{i})^2(z+\mathrm{i})^2}.$$

For R large enough ($R > 1$), Γ contains exactly one second order pole $z_0 = \mathrm{i}$ of $f(z)$. Since i is a second order pole, we have

$$\mathrm{Res}(f,\mathrm{i}) = \lim_{z\to\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z}[(z-\mathrm{i})^2 f(z)] = \lim_{z\to\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}z}\frac{z+1}{(z+\mathrm{i})^2} = \lim_{z\to\mathrm{i}} \frac{(z+\mathrm{i})^2 - 2(z+1)(z+\mathrm{i})}{(z+\mathrm{i})^4} = -\frac{\mathrm{i}}{4}.$$

Therefore, for $R > 1$ we have

$$\int_\Gamma f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\mathrm{Res}(f,\mathrm{i}) = \frac{\pi}{2}.$$

Then

$$\frac{\pi}{2} = \lim_{R\to\infty} \int_{\gamma_{-R,R}} f(z)\,\mathrm{d}z + \lim_{R\to\infty} \int_{C_R^+} f(z)\,\mathrm{d}z.$$

Since $f(z) = P(z)/Q(z)$ with $\deg Q = 4 \geq 3 = \deg P + 2$ we know that the second limit is 0. The first integral is I so we have

$$I = \frac{\pi}{2}.$$

Question 3 (20 points)

Consider the function

$$f(z) = z e^{\mathrm{i}/z^2}.$$

The Laurent series of $f(z)$ for $|z| > 0$ is

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n,$$

where it is given that $c_0 = 0$, $c_1 = 1$, and $c_{-1} = \mathrm{i}$.

(a) (8 points) Compute the rest of the coefficients of the Laurent series of $f(z)$ for $|z| > 0$.

Solution

Since $|z| > 0$, write $w = 1/z$ and consider the function

$$f(1/w) = \frac{1}{w} e^{\mathrm{i}w^2}.$$

The Taylor series at 0 for $e^{\mathrm{i}w^2}$ is

$$e^{\mathrm{i}w^2} = 1 + \mathrm{i}w^2 + \frac{1}{2}\mathrm{i}^2 w^4 + \frac{1}{3!}\mathrm{i}^3 w^6 + \cdots = \sum_{k=0}^{\infty} \frac{\mathrm{i}^k}{k!} w^{2k}.$$

Therefore, the Laurent series for $f(1/w)$ is

$$f(1/w) = \frac{1}{w} e^{\mathrm{i}w^2} = \frac{1}{w} + \mathrm{i}w + \frac{1}{2}\mathrm{i}^2 w^3 + \frac{1}{3!}\mathrm{i}^3 w^5 + \cdots = \sum_{k=0}^{\infty} \frac{\mathrm{i}^k}{k!} w^{2k-1}.$$

Writing $w = 1/z$ we find

$$f(z) = z + \frac{\mathrm{i}}{z} + \frac{\mathrm{i}^2}{2z^3} + \frac{\mathrm{i}^3}{3!z^5} + \cdots = \sum_{k=0}^{\infty} \frac{\mathrm{i}^k}{k! z^{2k-1}}.$$

Therefore, $c_n = 0$ for $n \geq 2$, $c_1 = 1$, $c_0 = 0$, $c_n = 0$ for n even and negative and $c_n = \mathrm{i}^k/k!$ for $n = -2k + 1$, $k \geq 0$.

(b) (4 points) Give the type of the singularity of f at $z_0 = 0$ (removable, pole of order m, essential). Justify your answer.

Solution

Since the Laurent series contains infinitely many terms with $c_n \neq 0$, $n \leq 0$ we conclude that this is an essential singularity.

(c) (4 points) Determine the residue of f at $z_0 = 0$. Justify your answer.

Solution

By definition, $\mathrm{Res}(f, 0) = c_{-1} = \mathrm{i}$.

(d) (4 points) Determine the domain in which the Taylor series of $f(z)$ at $z_1 = 1+\mathrm{i}$ converges. Justify your answer.

Solution

The only singularity of $f(z)$ is $z_0 = 0$. The distance of z_1 from z_0 is

$$R = |z_1 - z_0| = |1 + \mathrm{i}| = \sqrt{2}.$$

Therefore the Taylor series at $z_1 = 1 + \mathrm{i}$ converges in the open disk

$$D = \{|z - (1 + \mathrm{i})| < \sqrt{2}\}.$$

Question 4 (15 points)

(a) (6 points) Let m be a positive integer. Show that for all z on the unit circle we have

$$\left| 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots + \frac{z^m}{m!} \right| < e.$$

Solution

By the triangle inequality, we have for $|z| = 1$ that

$$\left|1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots + \frac{z^m}{m!}\right| \leq 1 + |z| + \frac{|z|^2}{2} + \frac{|z|^3}{3!} + \cdots + \frac{|z|^m}{m!}$$
$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{m!} = \sum_{k=0}^{m} \frac{1}{k!} < \sum_{k=0}^{\infty} \frac{1}{k!} = e.$$

(b) (9 points) Let m, n be positive integers. Show that the polynomial

$$P(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots + \frac{z^m}{m!} + 3z^n$$

has exactly n zeros inside the unit disk. Give complete arguments.

Solution

Let $f(z) = 3z^n$ and $h(z) = 1 + z + \cdots + z^m/m!$. Both functions are analytic on and inside the unit circle and on the unit circle we have $|f(z)| = 3$. Therefore for $|z| = 1$ we have

$$|h(z)| < e < 3 = |f(z)|.$$

Therefore, the conditions for applying Rouché's theorem hold and we have

$$N_0(P) = N_0(f) = n,$$

since $f(z) = 3z^n$ has a zero of multiplicity n at $z_0 = 0$ (inside the unit disk).

Question 5 (15 points)

Compute the following integrals along the path γ shown below that lies in the left half-plane, starts at πi and ends at -2πi. Give complete arguments.

(a) (6 points) $\int_\gamma z \, \mathrm{d}z$.

Solution

We have $(z^2/2)' = z$. Therefore,

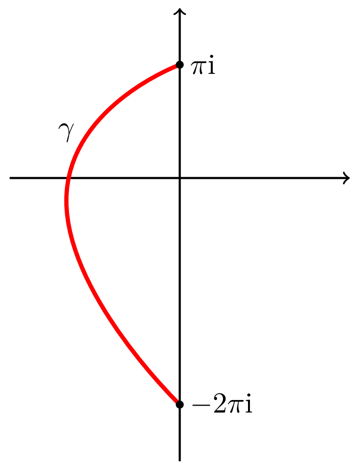
$$\int_\gamma z \, \mathrm{d}z = \frac{(-2\pi \mathrm{i})^2}{2} - \frac{(\pi \mathrm{i})^2}{2} = \frac{-4\pi^2}{2} - \frac{-\pi^2}{2} = -\frac{3\pi^2}{2}.$$

(b) (9 points) $\int_\gamma \frac{1}{z} \, \mathrm{d}z$.

Solution

We have $L'(z) = 1/z$ where $L(z)$ is the branch of the logarithm obtained by taking the argument in the interval $[0, 2\pi)$ (and thus it has a branch cut along the positive real axis). Then

$$\int_\gamma \frac{1}{z} \, \mathrm{d}z = L(-2\pi \mathrm{i}) - L(\pi \mathrm{i}) = \mathrm{Log}\,|-2\pi \mathrm{i}| + \mathrm{i}A(-2\pi \mathrm{i}) - \mathrm{Log}\,|\pi \mathrm{i}| - \mathrm{i}A(\pi \mathrm{i})$$
$$= \mathrm{Log}(2\pi) + \mathrm{i}\frac{3\pi}{2} - \mathrm{Log}(\pi) - \mathrm{i}\frac{\pi}{2} = \mathrm{Log}\,2 + \pi \mathrm{i}.$$



Question 6 (10 points)

Answer only one of the following two questions:

Question A. Consider the Möbius transformation

$$f(z) = \frac{2z}{z+1}$$

on the extended complex plane $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. After computing $f(0)$, $f(\pm 1)$, and $f(\pm \mathrm{i})$, determine the image of the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$ under f.

Solution

We first check the images of the given points. We compute

$$f(0) = 0, \; f(1) = 1, \; f(-1) = \infty, \; f(\mathrm{i}) = 1 + \mathrm{i}, \; f(-\mathrm{i}) = 1 - \mathrm{i}.$$

Therefore, the unit circle is mapped to the straight line going through the points 1, $1 \pm \mathrm{i}$, that is, the line $\ell = \{z \in \mathbb{C} : \operatorname{Re} z = 1\}$.

Moreover, since 0 is mapped to 0 we conclude that the closed unit disk is mapped to the set $\{z \in \mathbb{C} : \operatorname{Re} z \leq 1\}$.

Question B. Given that $f(z)$ is analytic at $z = 0$ and that $f(1/n) = 1/n^4$ for $n = 1, 2, \ldots$, find $f(z)$. Justify your conclusion that your solution $f(z)$ is the *only* function satisfying these properties. Then find a function $g(z)$ which is *not* analytic at $z = 0$ and satisfies $g(1/n) = 1/n^4$ for $n = 1, 2, \ldots$. *Hint:* Find a function $h(z)$ which is not analytic at $z = 0$ and satisfies $h(1/n) = 0$ for $n = 1, 2, \ldots$.

Solution

The required function is $f(z) = z^4$ since then we have $f(1/n) = 1/n^4$ and $f(z)$ is analytic at $z = 0$. The uniqueness follows from the fact that two analytic functions that agree on a convergent sequence of point are identical.

Let $h(z) = \sin(\pi/z)$. Then $h(1/n) = \sin(n\pi) = 0$ but the function is not analytic at $z = 0$. Then define

$$g(z) = f(z) + h(z) = z^4 + \sin(\pi/z).$$

The given g satisfies $g(1/n) = f(1/n) + h(1/n) = 1/n^4$ and it is not analytic at $z = 0$.

Formulas

The Cauchy-Riemann equations for a function $f = u + \mathrm{i}v$ are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

The principal value of the logarithm is

$$\operatorname{Log} z = \operatorname{Log}|z| + \mathrm{i}\operatorname{Arg} z.$$

The generalized Cauchy integral formula is

$$f^{(n)}(z_0) = \frac{n!}{2\pi \mathrm{i}} \int_\Gamma \frac{f(z)}{(z - z_0)^{n+1}}\,\mathrm{d}z.$$

The residue of a function f at a pole z_0 of order m is given by

$$\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} \left[(z - z_0)^m f(z)\right].$$